Digital Signal Processing Tutorial

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Contents:

- Digital signals, sampling, and spectra
- Filtering

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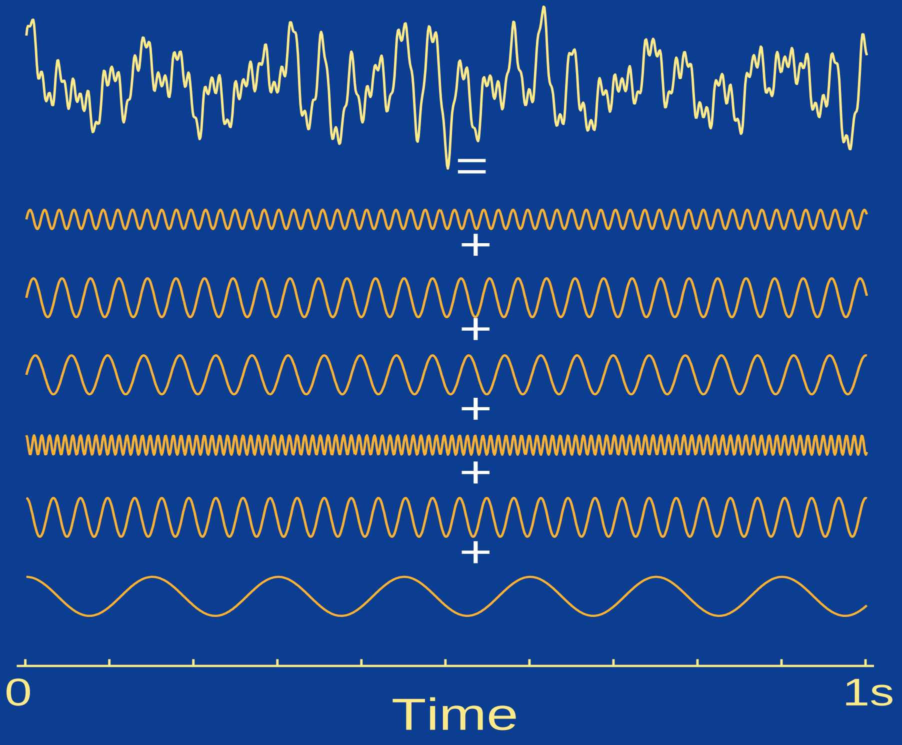
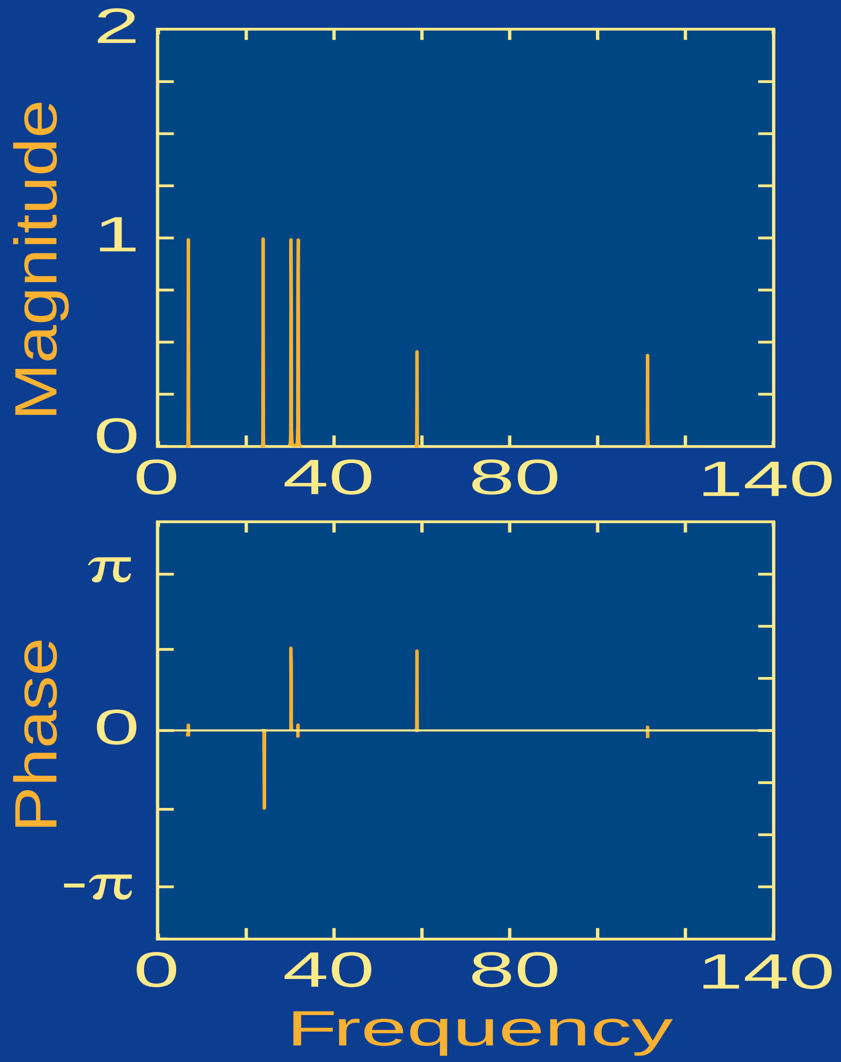
Spectral decomposition





- All signals can be decomposed to sinusoidals

Fourier Transform

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{i2\pi f t} dt$$

Fourier transorm

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{-i2\pi f t} df$$

Inverse Fourier transform

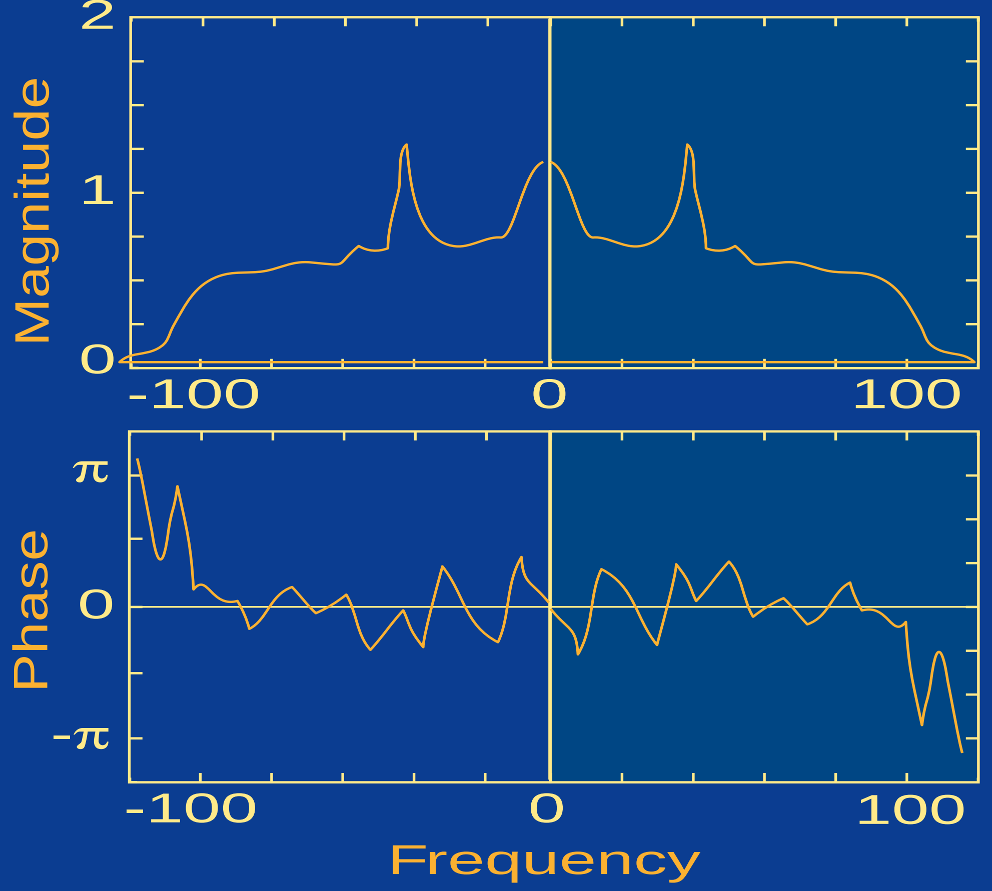
$$h(t) = \int_{-\infty}^{\infty} A(f) \cos\left(2\pi f t + \theta(f)\right) df$$

Real part of the signal

$$-i \int_{-\infty}^{\infty} A(f) \sin\left(2\pi f t + \theta(f)\right) df$$

Imaginary part

One sided spectrum



For real valued signals

$$A(f) = A(-f)$$

$$\theta(f) = -\theta(-f)$$

The negative side does not carry any new information

Power spectral density

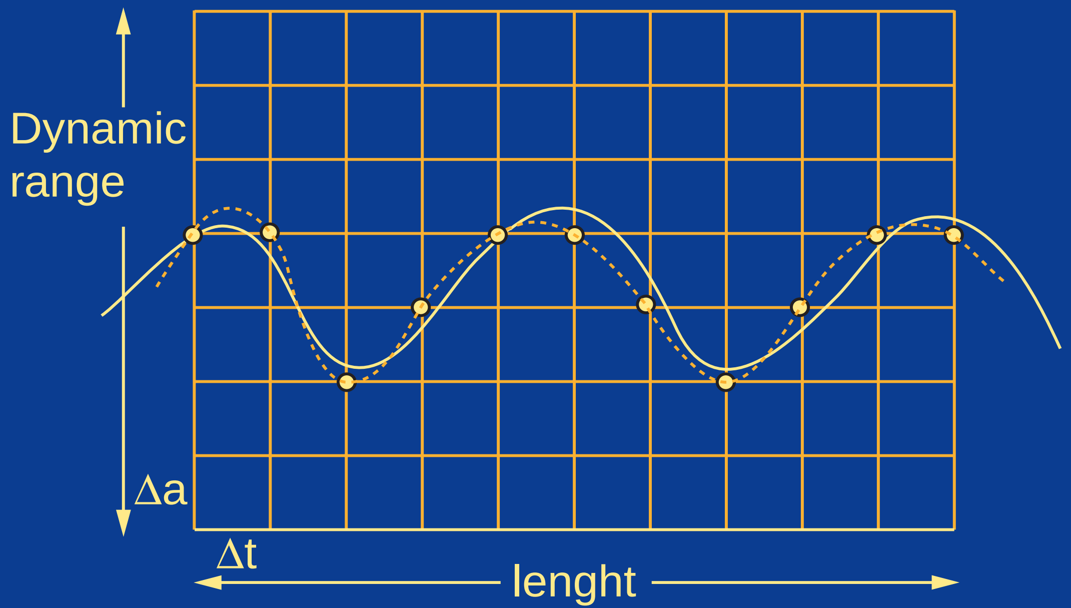
Power spectral density

- Two sided: $PSD = \left|H(f)\right|^2$
- Onesided: $PSD = \left|H(f)\right|^2 + \left|H(-f)\right|^2$
- Unit e.g. $\dfrac{T^2}{Hz}$

RMS spectral density

- $SD = \sqrt{\left|H(f)\right|^2 + \left|H(-f)\right|^2}$
- Unit e.g. $\dfrac{T}{\sqrt{Hz}}$

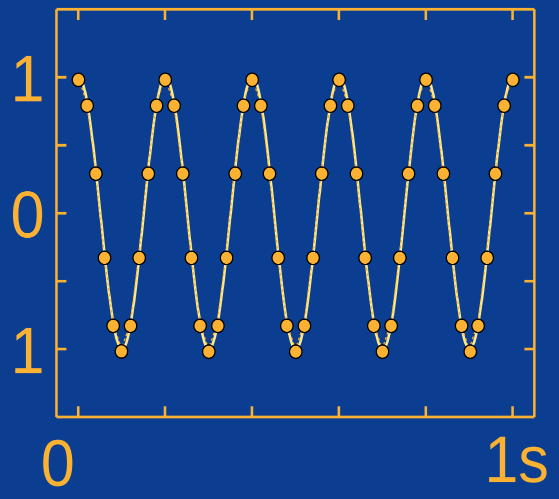
Sampling continuous signals

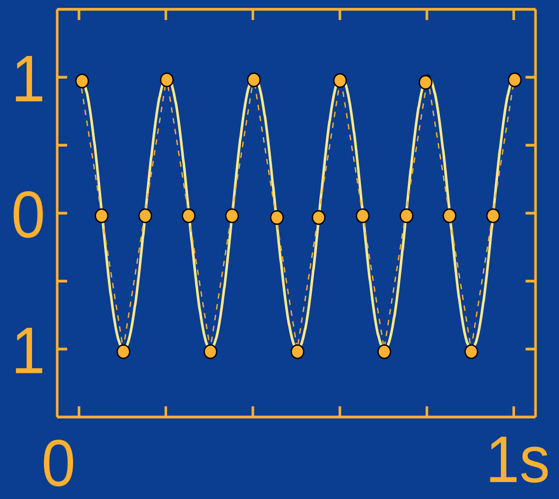


Δa = amplitude resolution
Δt = time resolution

- Amplitude resolution
- Dynamic range
- Time resolution
- Length in time

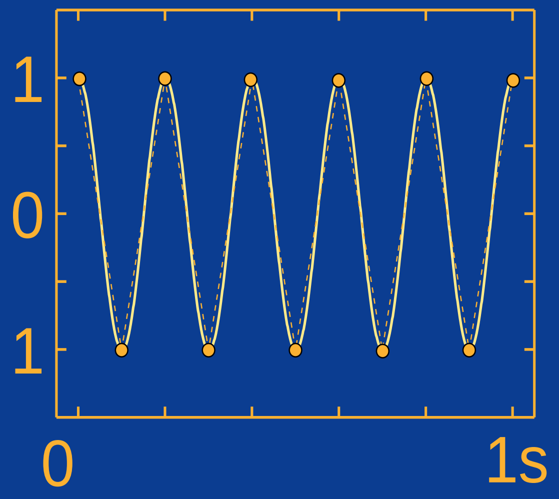
Nyquist frequency

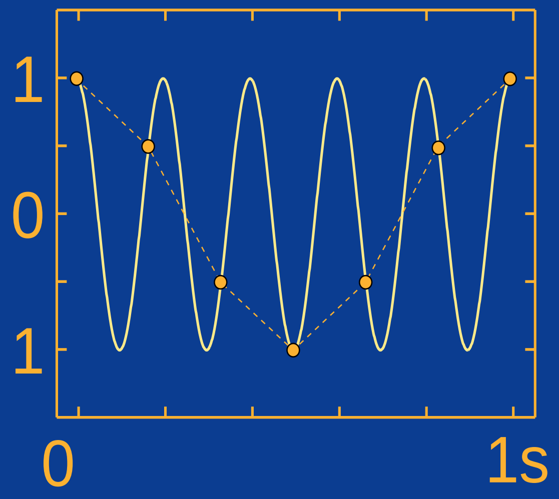




Nyquist frequency

Maximum allowed frequency $f_n = 1/2\, f_s$

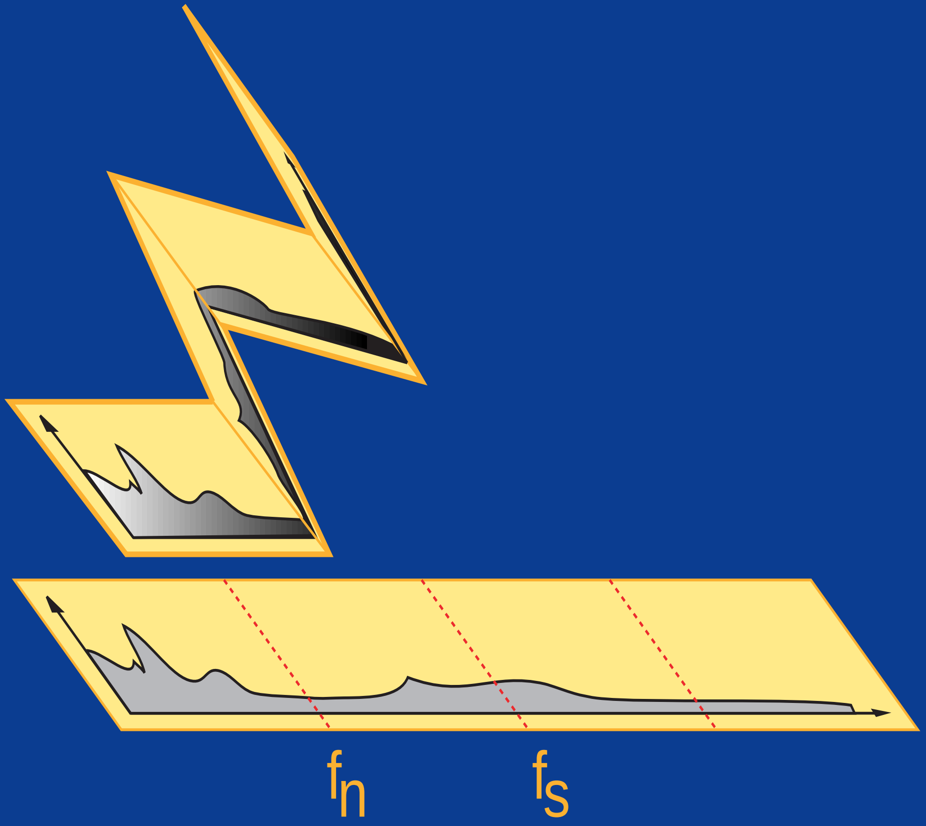




Aliasing

If $f > f_n$ it appears as a lower frequency.

Aliasing / Folding



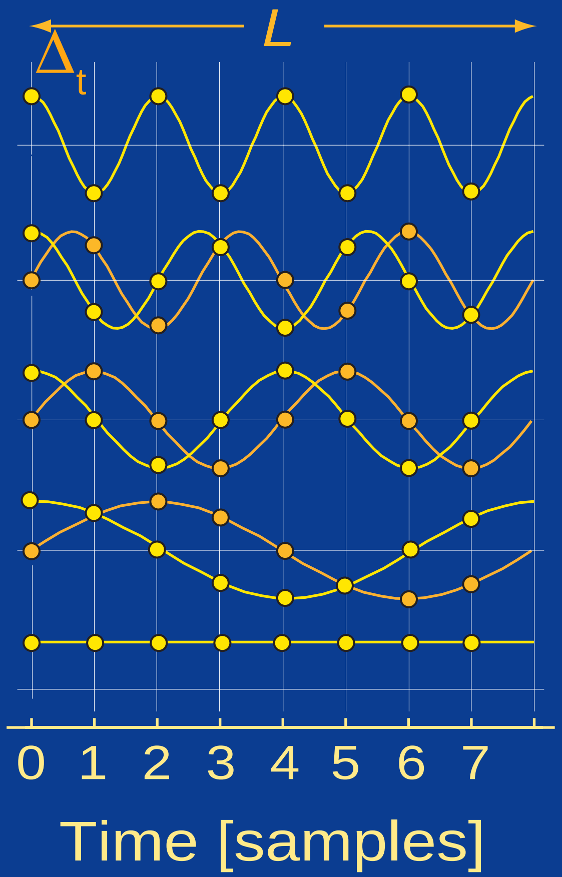
Due to the way how the too high frequencies are aliased, this is also called "folding".

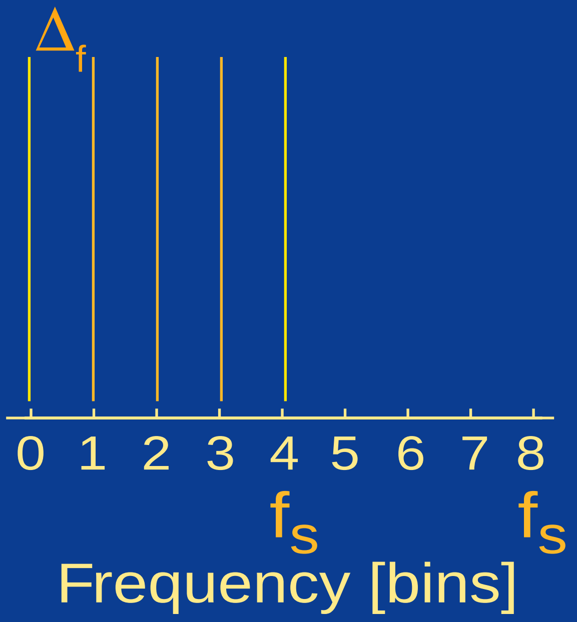
Discrete Fourier Transformation

$$H_n = \sum_{k=0}^{N-1} h_k \, e^{i(\frac{2\pi}{N})nk}$$

$$h_k = \frac{1}{N}\sum_{n=0}^{N-1} H_n \, e^{-i(\frac{2\pi}{N})nk}$$

$$H(f_n) \approx \Delta_t H_n$$

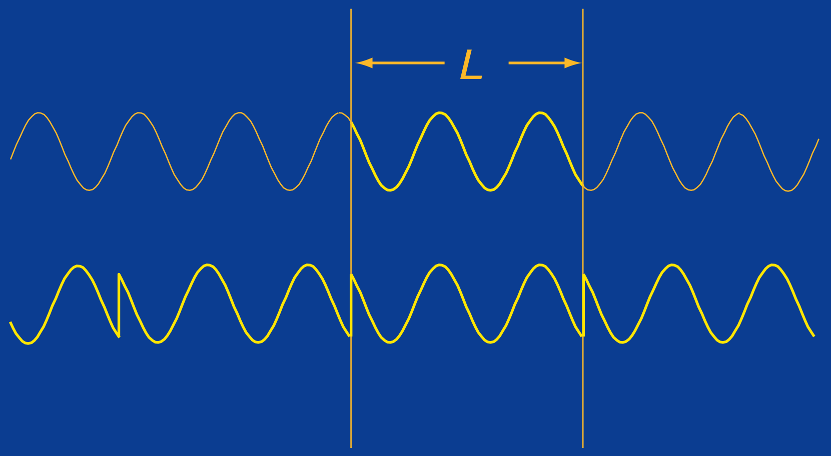




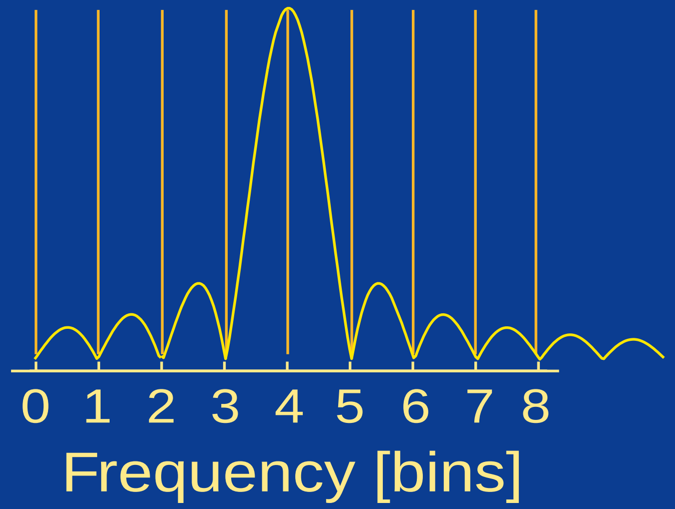
$$\Delta_f = \frac{f_s}{N} = \frac{1}{L}$$

$$N_{bins} = \left(\frac{N}{2} + 1\right)$$

Leakage

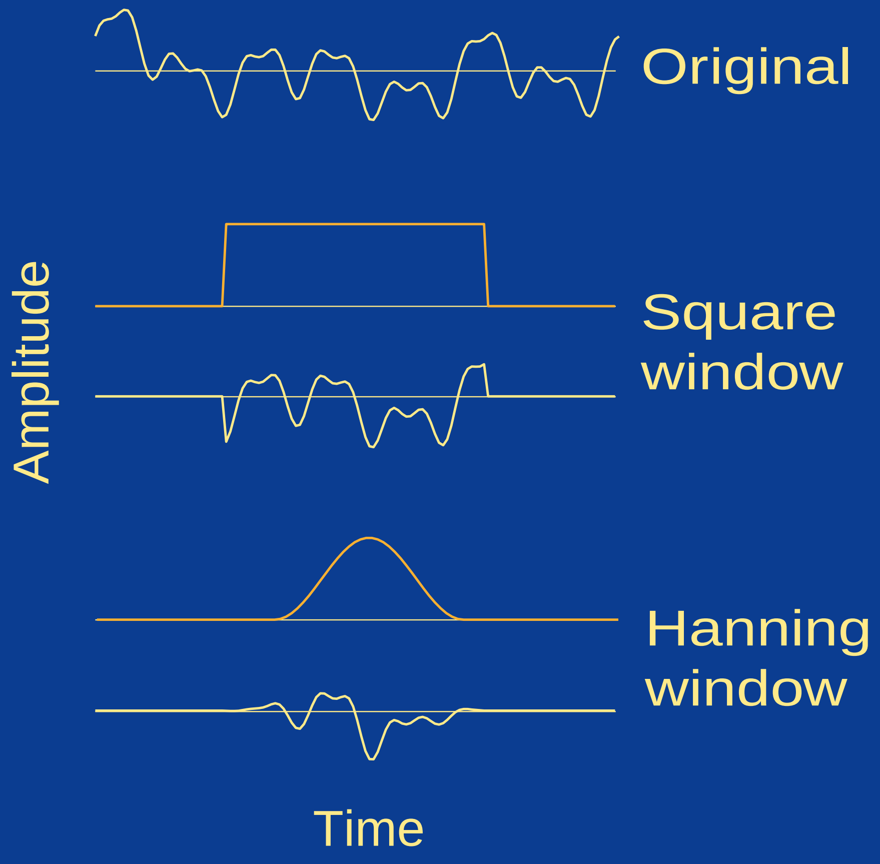


DFT assumes a cyclic extension of the data.

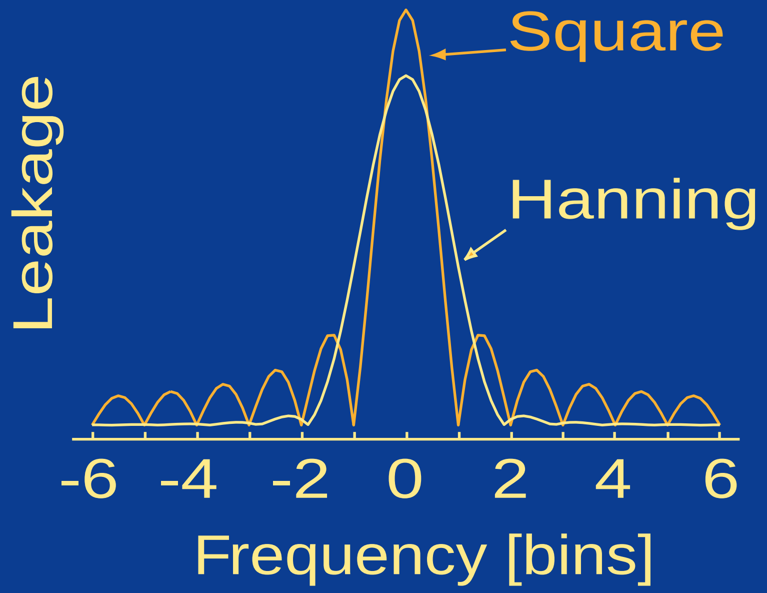


Frequencies between bin centers "leak" also beyond nearest bins

Data Windowing



Windowing in time domain



Windowing

- Decreases side lobes
- Widens the main lobe

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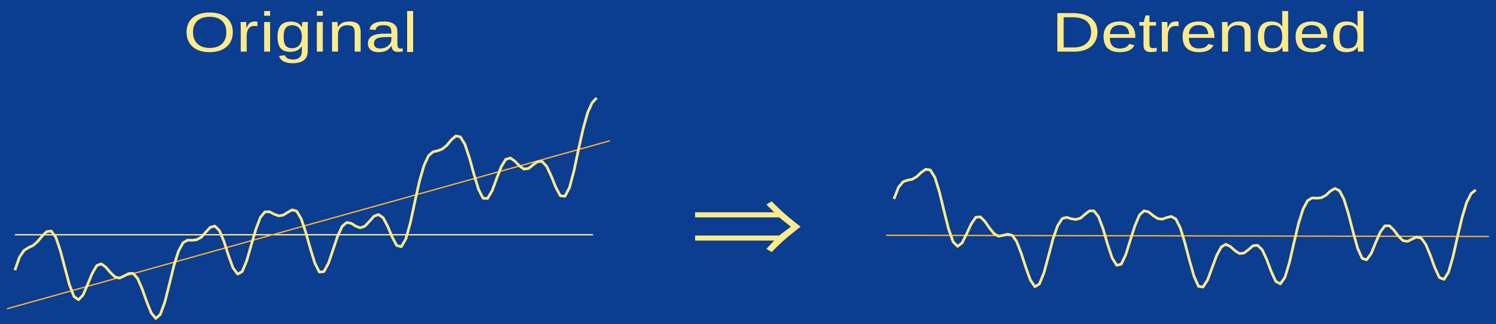
Detrending



Original

⇒

Detrended

- Fit a polynomial of low degree (typically a line)
- Substract the polynomial

Averaging Spectra

PSD estimate at each bin

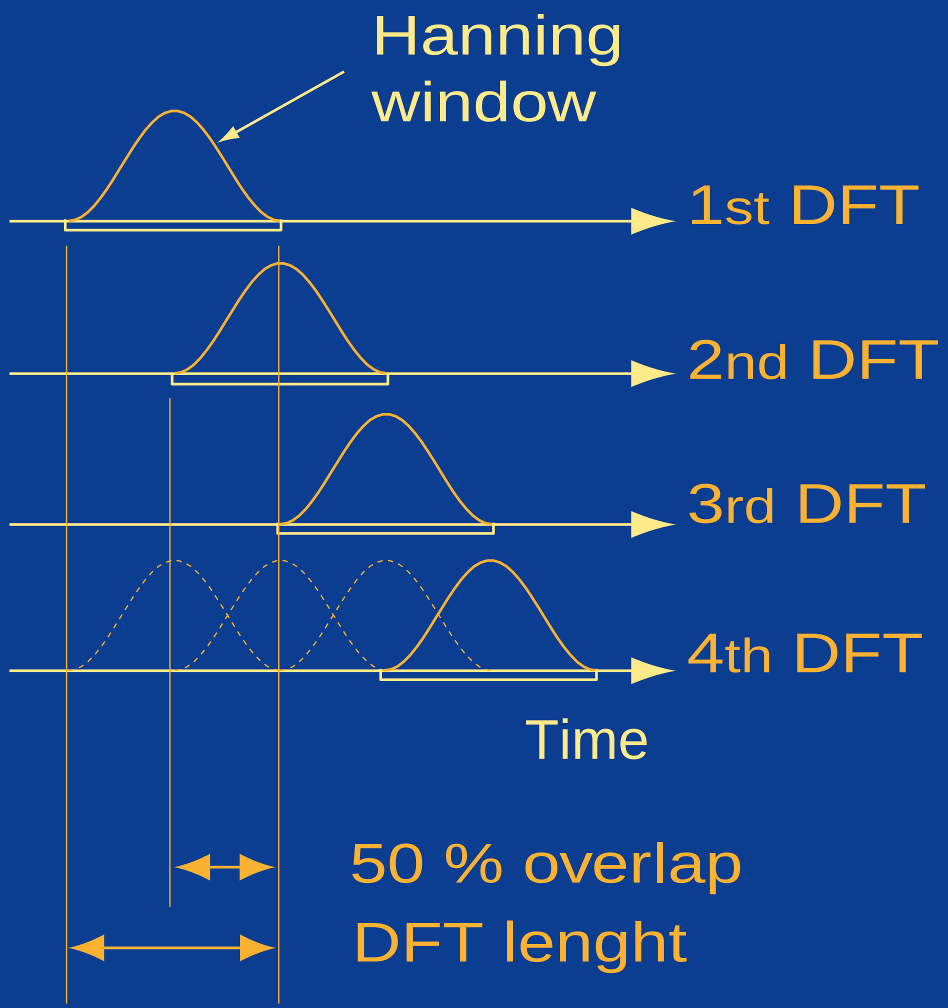
- Expection value $E\{P\}$ = Mean power in "window"
- Standard deviation $\sigma = E\{P\}$

How to decrease the variance?

- Increasing DFT length does not help!
- Shorter transforms & averaging
- Smoothing the spectrum (combining bins together)

Bottom line: $\sigma \sim \sqrt{\dfrac{Number\,of\,bins}{Number\,of\,samples}}$

Averaging Spectra



The most common way:

- Split data span into overlapping segments
- Apply window to each segment
- Apply DFT to each segment
- Square
- Average spectra

FFT

DFT

- (Mathematical) transformation
- Can be calculated to any data length N

FFT

- Algorithm to calculate DFT efficiently
- Requires length $N = 2^n$
- Number of operations needed $\sim N \log_2(N)$ compared to N^2

Autoregressive method

Alias

- Maximum entropy method
- All poles method

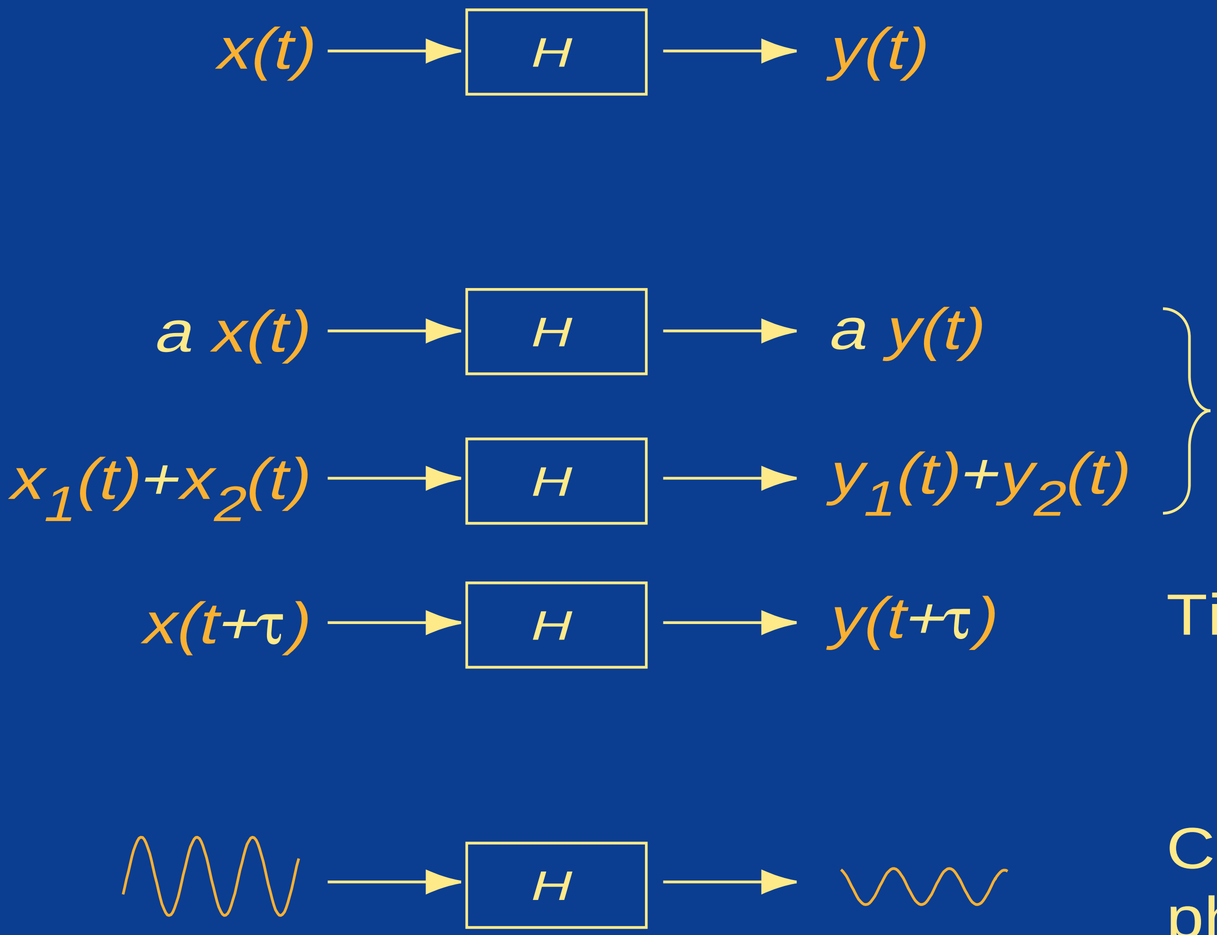
Steps:

- Estimate autocorrelation (non-cyclical)
- Fit an autoregressive model to this.
- Convert model to frequncy domain.

Features

- No cyclical extension
- Can represent infinite power density
- Parametric, correct order?

Linear filters



$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$

$a\,x(t) \longrightarrow \boxed{H} \longrightarrow a\,y(t)$

$x_1(t)+x_2(t) \longrightarrow \boxed{H} \longrightarrow y_1(t)+y_2(t)$

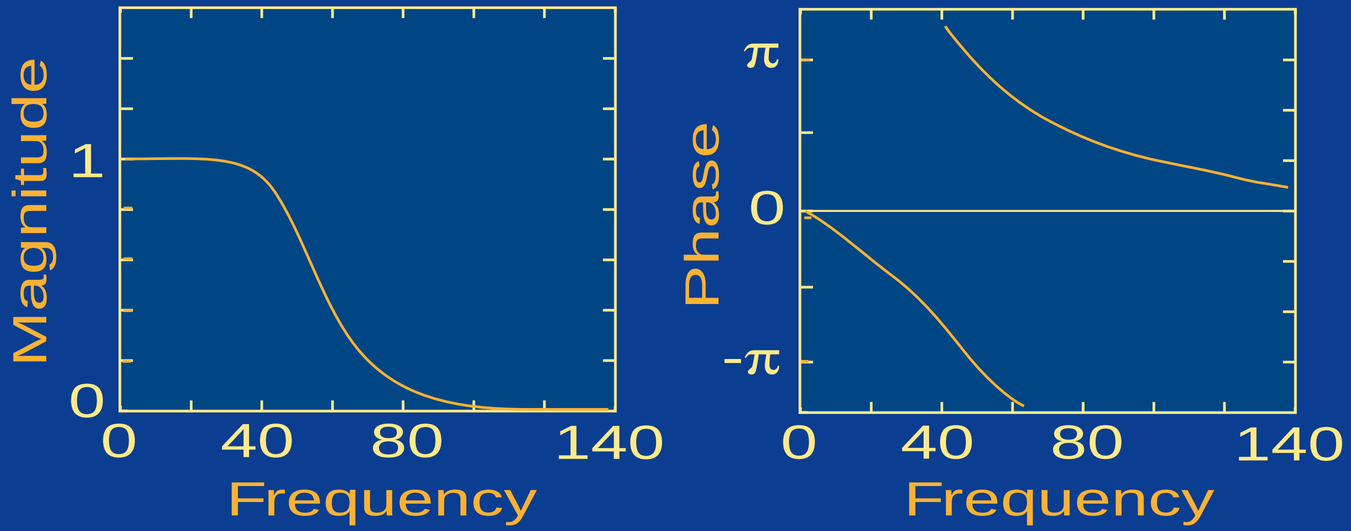
Linear

$x(t+\tau) \longrightarrow \boxed{H} \longrightarrow y(t+\tau)$

Time invariant

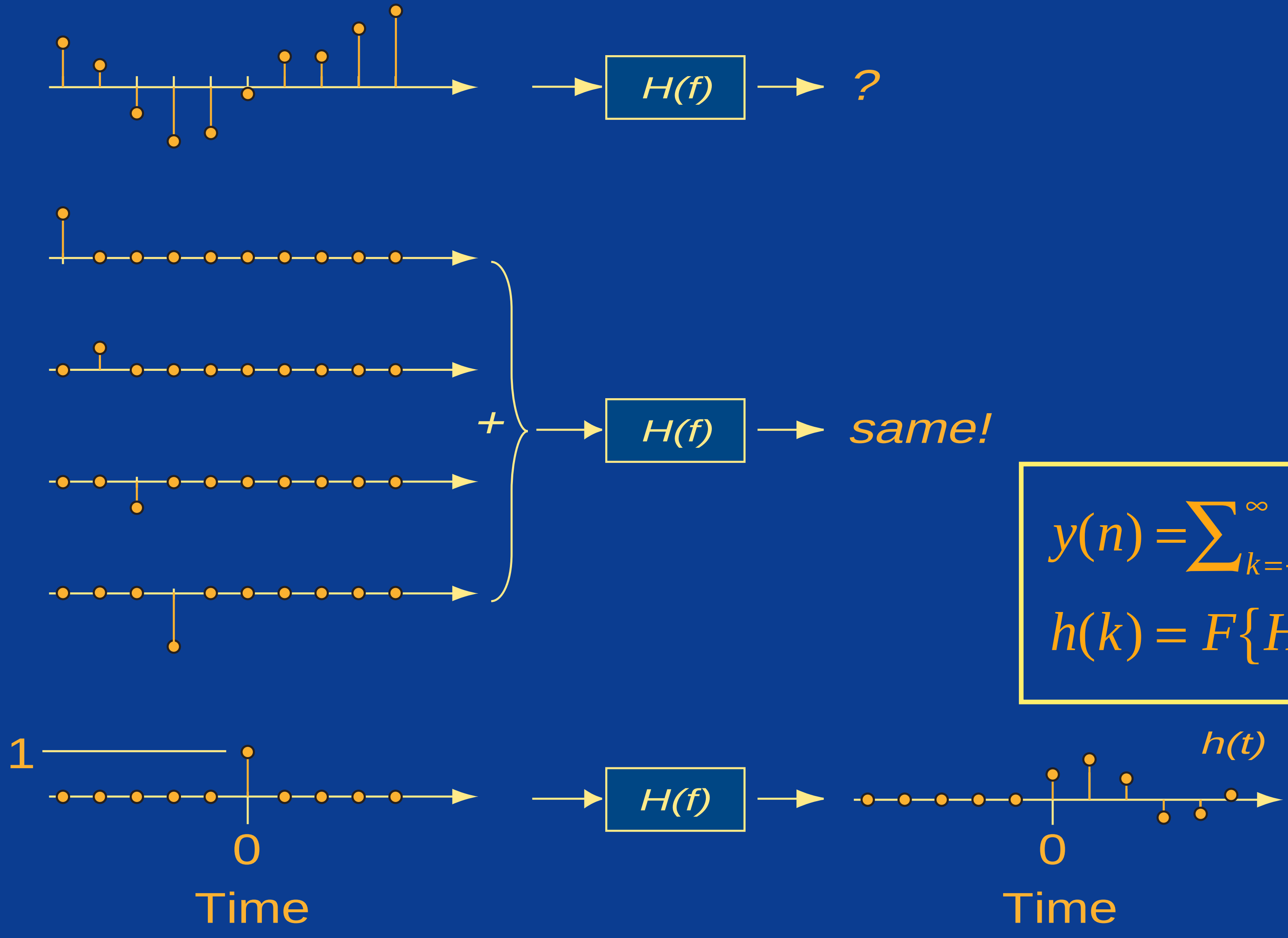
Changes amplitude and phase but NOT frequency.

Frequency response



Frequency response of a 4th order Butterworth filter

Impulse Response



$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$h(k) = F\{H(n)\}$$

Causal/Non-causal filters